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Dependence of Alternants on Functions

CHARLES B. DUNHAM

Computer Science Department, The University of Western Ontario, London, Ontario, Canada N6A 5B7

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Alternation based algorithms (e.g., Remez' 2nd algorithm) for best Chebyshev approximation depend on elements of trial alternants being well separated (in Remez' case this helps guarantee not too badly conditioned [11] linear systems of equations associated with the Remez levelling equations). In non-degenerate cases, a separation exists for alternants of nearby functions, but in ordinary best rational approximation a coalescence of elements of alternants is exhibited in "near-degenerate" cases, suggesting difficulties for alternation based algorithms.

Let $[\alpha, \beta]$ be a finite interval and $\| \ \|$ the Chebyshev norm on $C[\alpha, \beta]$. Consider approximation by an alternating family [4; 8, p. 15ff] $\{F(A, \cdot): A \in P\}$, that is, a subset of $C[\alpha, \beta]$ such that $F(A, \cdot)$ possesses a *degree* $\rho(A)$ so that $F(A, \cdot)$ is the best approximation to f if and only if $f - F(A, \cdot)$ alternates $\rho(A)$ times on $[\alpha, \beta]$. Denote the best approximation to f (if it exists) by Tf.

DEFINITION. A function g alternates l times on $C[\alpha, \beta]$ if there exists $\{x_0, ..., x_l\}$, where $\alpha \leqslant x_0 < \cdots < x_l \leqslant \beta$, such that

$$|g(x_i)| = \|g\|$$
$$g(x_i) = (-1)^i g(x_0) \qquad i = 0, ..., l.$$

The set $\{x_0, ..., x_l\}$ is called an *alternant* of g. The problem we consider is the dependence of alternants of $f - Tf$ on f.

The following definition is taken from [4].

DEFINITION. F is *degenerate* at A if every neighborhood of $F(A, \cdot)$ contains an element of (F, P) of higher degree.

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THEOREM. *Let Tf be non-degenerate and of degree n. Let $\{f_k\} \to f$ and $\{x_0^k, ..., x_n^k\}$ be alternants of $f_k - Tf_k$. Any accumulation point of $(x_0^k, ..., x_n^k)$ is an alternant of $f - Tf$.*

Proof. By Theorem 8 [4, p. 106], $\{Tf_k\} \to Tf$ uniformly, hence by definition of degeneracy and the second corollary to Theorem 2 of [4], Tf_k is of degree n also for k sufficiently large, hence alternants are of the correct length. Assume $\{x_i^k\} \to x_i^0$, $i = 0, ..., n$. Now suppose that for some i and for some $\varepsilon > 0$ $|(f - Tf)(x_i^0)| < \|f - Tf\| - \varepsilon$. Then there is a neighbourhood N of x_i^0 such that

$$|(f - Tf)(y)| < \|f - Tf\| - \varepsilon, \qquad y \in N.$$

It follows that for all k sufficiently large,

$$|(f_k - Tf_k)(y)| < \|f - Tf\| - \varepsilon/2 \qquad y \in N.$$

But $\|f_k - Tf_k\| \to \|f - Tf\|$ and $x_i^k \in N$ for all k sufficiently large, contradicting x_i^k being an extremum. Let $(f - Tf)(x_0^0) = \sigma \|f - Tf\|$ then for $i = 0, ..., n$ $(f_k - Tf_k)(x_i^k) = \sigma(-1)^i \|f_k - Tf_k\|$ for $\sigma = +$ or $-$, respectively, for all k sufficiently large. Hence $f - Tf$ alternates on $\{x_0^0, ..., x_n^0\}$.

Remark. If (F, P) is varisolvent and Tf is of maximum degree, Tf_k must exist for all k sufficiently large [3].

For the rest of this paper consider a special case, approximation by ordinary rational functions $R_m^n[\alpha, \beta]$:

$$R(A, x) = P(A, x)/Q(A, x) = \sum_{k=0}^{n} a_k x^k \Big/ \sum_{k=0}^{m} a_{n+1+k} x^k.$$

Best approximation was first characterized by Achieser [1; 7, p. 80; 9] in terms of alternation and defect (= degeneracy) $d(r) = \min\{n - \partial P(A, x), m - \partial(A, x)\}$.

THEOREM. *R is best to f if and only if $f - R$ alternates $n + m + 1 - d(R)$ times on $[\alpha, \beta]$.*

LEMMA. *Suppose $\{A^k\} \to A^0$ with $Q(A^0, \cdot) \geq 0$ on $[\alpha, \beta]$, and reducing $P(A^0, \cdot)/Q(A^0, \cdot)$ (if necessary) yields an element $p/q \in R_m^n[\alpha, \beta]$ with defect 1. Then there is a subsequence $\{A^{k(j)}\}$ and an endpoint of $[\alpha, \beta]$ such that $\{R(A^{k(j)}\}$ converges uniformly to p/q on any closed subset of $[\alpha, \beta]$ excluding this endpoint.*

Proof. If $Q(A^0, \cdot)$ has no zeros on $[\alpha, \beta]$ uniform convergence of the full sequence follows on $[\alpha, \beta]$. Assume $Q(A^0, \cdot)$ has a zero. As p is of exact degree $n-1$ or q is of exact degree $m-1$, $P(A^0, \cdot)$ and $Q(A^0, \cdot)$ can only have a polynomial of degree 1 as a common factor. It must in fact be either $(x-\alpha)$ or $(\beta - x)$ for if $(x-\gamma)$ were a factor for γ interior, $(x-\gamma)q$ would not be $\geqslant 0$ on $[\alpha, \beta]$ and if $(x-\gamma)$ were a factor for γ exterior, $(x-\gamma)q$ would have no zeros on $[\alpha, \beta]$. Assume $(x-\alpha)$ is the common factor, then $Q(A^0, \cdot) > 0$ on $[\gamma, \beta]$ for any $\gamma > \alpha$, hence $R(A^k, \cdot) \to P(A^0, \cdot)/Q(A^0, \cdot) = p/q$ uniformly on $[\gamma, \beta]$.

THEOREM. *Let Tf have defect* 1. *Let $f - Tf$ have precisely $n+m+1$ extrema and endpoints be extrema. Let $\{f_k\} \to f$ and Tf_k be non-degenerate. Let $\{x_0^k, ..., x_{n+m+1}^k\}$ be an alternant of $f_k - Tf_k$. Then there exists a subsequence of $\{f_k\}$ such that the corresponding alternants have two points tending to an endpoint.*

Proof. Let $Tf_k = R(A^k, \cdot)$ then by [6], $\{A^k\}$ has an accumulation point A^0 and $P(A^0, \cdot)/Q(A^0, \cdot)$ is best to f under the constraint $Q(A, \cdot) \geqslant 0$, hence by reducing $P(A^0, \cdot)/Q(A^0, \cdot)$ if necessary yields Tf. By the lemma we can assume without loss of generality that $\{R(A^k, \cdot)\} \to Tf$ uniformly on $[\gamma, \beta]$ for any $\gamma > \alpha$. If the theorem were false, there would exist $\delta > 0$ such that $\alpha + \delta < x_1^k$ for all k sufficiently large. Select $\gamma < \delta/2$ then $\{f_k - Tf_k\} \to \{f - Tf\}$ uniformly on $[\alpha + \delta, \beta]$. As $f_k - Tf_k$ alternates $n+m$ times on $[\alpha + \delta, \beta]$ with amplitude $\|f_k - Tf_k\|$, $f - Tf$ alternates $n+m$ times on $[\alpha + \delta, \beta]$ with amplitude $\|f - Tf\|$. But this contradicts our hypothesis on extrema of $f - Tf$.

Remark. Coalescing of extrema leads to failure in alternation based algorithms such as the second algorithm of Remez [8, p. 105ff] and Maehly's second method [8, p. 113ff].

Remark. It is clear from the theorem that $\{Tf_k\} \not\to Tf$ uniformly on $[\alpha, \beta]$, which also follows from [10, p. 324].

In view of the relationship between degeneracy, separation in alternants, and uniform strong uniqueness (SU) constants [12], the question of a possible uniform SU constant for the sequence $\{f_k\}$ of the last theorem might be raised. The answer is that any sequence $\{g_k\}$ which has an accumulation point f with a degenerate best approximation $\neq f$ (so that the SU constant for f must be zero, a consequence of the discontinuity result of Werner [9] and strong uniqueness implying a Lipschitz constant [2, p. 82]) cannot have a uniform SU constant [10, Section 4].

Remark. If an endpoint were not an extremum of $f - Tf$, drawing a diagram suggests that $f_k - Tf_k$ could have an additional extremum there and coalescing need not occur. In fact Cheney [2, p. 167] sketches construction of a set $\{f_\lambda\} \to f_0$ whose best approximation by $R_1^0[0, 1]$ is zero and $f_\lambda - Tf_\lambda$ has alternant $\{0, \frac{1}{2}, 1\}$.

In the discontinuity results of Werner [9], the requirement on the number of extreme points of $f - Tf$ is dropped, but only at the price of restricting attention to one sequence $\{f_k\} \to f$.

We have considered unweighted approximation, but arguments extend without change to weighted approximation if weights are positive and continuous (incorporate weights into f and R by multiplication).

The author is currently investigating approximation by powered rationals p^s/q^r, special cases of which were studied by Lau and by Kaufman and Taylor. Whether comparable results hold for these is open.

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